Problem 1.50

[Computer] The differential equation (1.51) for the skateboard of Example 1.2 cannot be solved in terms of elementary functions, but is easily solved numerically. **(a)** If you have access to software, such as Mathematica, Maple, or Matlab, that can solve differential equations numerically, solve the differential equation for the case that the board is released from $\phi_o = 20$ degrees, using the values $R = 5$ m and $g = 9.8$ m/s^2. Make a plot of ϕ against time for two or three periods. **(b)** On the same picture, plot the approximate solution (1.57) with the same $\phi_o = 20°$. Comment on your two graphs. Note: If you haven't used the numerical solver before, you will need to learn the necessary syntax. For example, in Mathematica you will need to learn the syntax for "NDSolve" and how to plot the solution that it provides. This takes a bit of time, but is something that is very well worth learning.

Solution

Equation (1.51) is on page 31.

$$\ddot{\phi} = -\frac{g}{R}\sin\phi \tag{1.51}$$

With $R = 5$ m and $g = 9.8$ m/s^2 and $\phi_o = 20°$, the initial value problem to solve is

$$\ddot{\phi} = -\frac{9.8}{5}\sin\phi, \quad \phi(0) = 20\left(\frac{\pi}{180}\right),\ \phi'(0) = 0$$

$$\ddot{\phi} = -1.96\sin\phi, \quad \phi(0) = \frac{\pi}{9},\ \phi'(0) = 0.$$

Note that $\phi(0) = \pi/9$ is the angle at $t = 0$, and $\phi'(0) = 0$ indicates that the particle starts from rest. To numerically solve this, type

$$\texttt{s} = \texttt{NDSolve}\left[\left\{\phi''[\texttt{t}] == -1.96\sin[\phi[\texttt{t}]],\ \phi[0] == \frac{\pi}{9},\ \phi'[0] == 0\right\}, \phi, \{\texttt{t}, 0, 14\}\right]$$

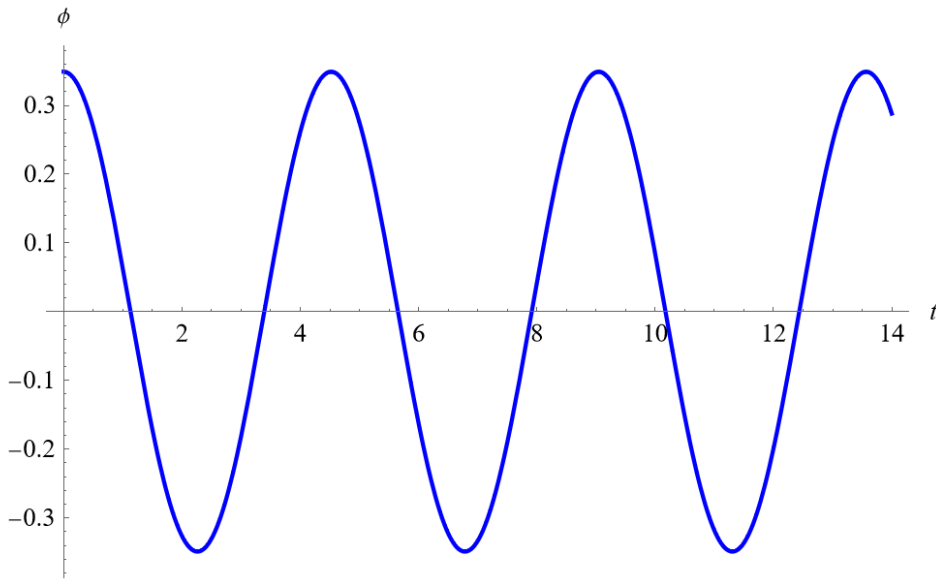
into Mathematica and press `Shift+Enter`. The output below is given as a result.

$$\left\{\left\{\phi \to \texttt{InterpolatingFunction}[\,]\right\}\right\}$$

In order to plot this function, type

$$\texttt{Plot}\left[\texttt{Evaluate}[\phi[\texttt{t}]\ /.\ \texttt{s}],\ \{\texttt{t}, 0, 14\},\ \texttt{PlotRange} \to \texttt{All},\ \texttt{AxesLabel} \to \{\texttt{t}, \phi\},\ \texttt{PlotStyle} \to \texttt{Blue}\right]$$

into Mathematica and press `Shift+Enter` to obtain the following graph.



By making the small-angle approximation, equation (1.51) becomes

$$\ddot{\phi} \approx -\frac{g}{R}\phi,$$

which has the exact solution,

$$\phi(t) = A\cos\left(\sqrt{\frac{g}{R}}\,t\right) + B\sin\left(\sqrt{\frac{g}{R}}\,t\right).$$

Apply the initial conditions to determine the constants, A and B.

$$\phi(0) = A = \frac{\pi}{9}$$

$$\phi'(0) = B\sqrt{\frac{g}{R}} = 0$$

Solving this system of equations yields $A = \pi/9$ and $B = 0$, which means

$$\phi(t) = \frac{\pi}{9}\cos\left(\sqrt{\frac{g}{R}}\,t\right).$$

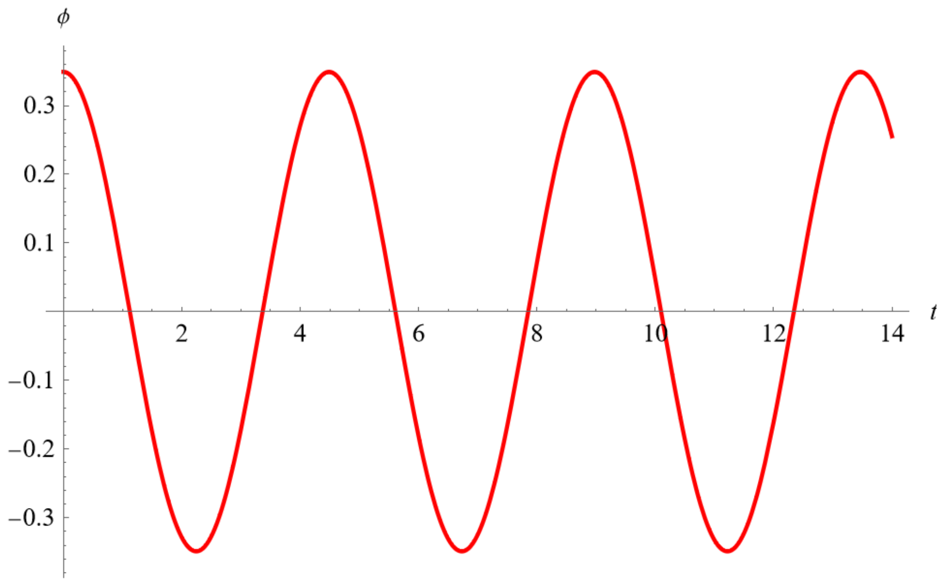
Therefore, with $R = 5$ m and $g = 9.8$ m/s^2,

$$\phi(t) = \frac{\pi}{9}\cos(1.4t).$$

In order to plot this function, type

$$\texttt{Plot}\left[\frac{\pi}{9}\cos[1.4\texttt{t}],\ \{\texttt{t}, 0, 14\},\ \texttt{PlotRange} \to \texttt{All},\ \texttt{AxesLabel} \to \{\texttt{t}, \phi\},\ \texttt{PlotStyle} \to \texttt{Red}\right]$$

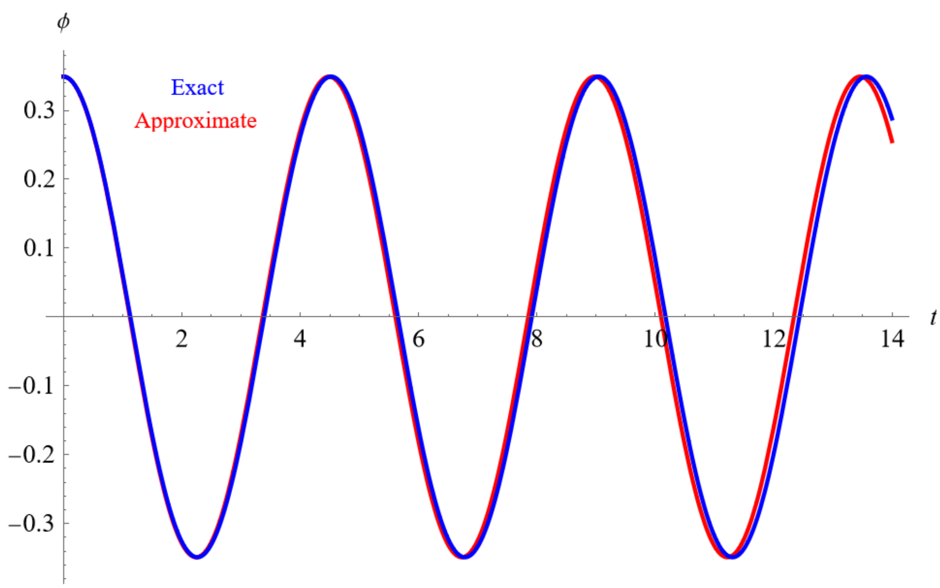
into Mathematica and press `Shift+Enter`.



To superimpose this graph with the previous one, type

$$\texttt{Show}\Big[\%, \%\%\Big]$$

into Mathematica and press `Shift+Enter`.



The graphs are indistinguishable for small times, but the difference between the graphs becomes more and more noticeable as time increases because the exact solution is not periodic.